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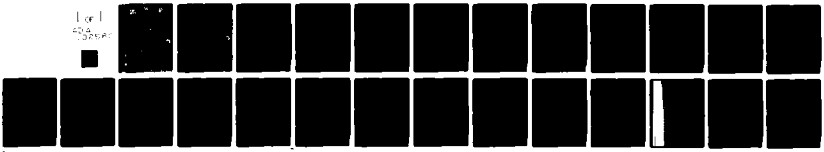

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Accelerated Life Tests Under Competing Weibull Causes of Failure

by

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ACCELERATED LIFE TESTS UNDER COMPETING WEIBULL CAUSES OF FAILURE

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Key Words and Phrases: Accelerated lifetests, competing risks, Weibull distribution, maximum likelihood.

ABSTRACT

Accelerated life testing of a product under more severe than normal condition is commonly used to reduce test time and cost. Data collected at such accelerated conditions is used to obtain estimates of parameters of a stress translation function which is then used to make inference about the products performance under normal conditions. This problem is considered when the product is a p component series system with Weibull distributed component lifetimes having a common shape parameter. A general stress translation function is used and estimates of model parameters are obtained for various censoring schemes.

1. INTRODUCTION

Accelerated life testing of a product is often used to obtain information on its performance under normal operating conditions. Such testing involves subjecting test items to conditions more severe than encountered in the item's everyday use. This results in decreasing the item's mean life and leads to reduced test time and experimental costs. In engineering applications accelerated conditions are produced by testing items at higher than normal temperature, load, voltage, pressure, etc., while in biological applications accelerated conditions arise from large doses of a chemical or radiological agents. In all cases the data collected at the high stresses is used to extrapolate to some low stress where testing is not feasiable.

We shall consider the problem of accelerated life testing when the item of interest is a p-component series system. Here the failure of any one of p components causes the system to fail. An observation on such a system consists of the system failure time and knowledge of which component's failure caused the system to fail. In the case where there is data only at operating conditions, David and Moeschberger (1978) describe some of the estimation techniques.

Several papers have been written on analyzing accelerated life tests for series systems. Assuming that for a given stress V each failure mode follows an independent log normal distribution with parameters $\mu i(v) = \alpha_i + \beta_i V$ and σ_i^2 constant with respect to V, i = 1, ..., P, Nelson (1973) obtains graphical estimates of α_i and β_i when there is no censoring. Maximum likelihood estimates of α_i, β_i and σ_i^2 are obtained in Nelson (1974).

Klein and Basu (1981a) have considered the above problem when the component lifetimes are exponentially distributed and the data is type I, type II or progressively censored. Klein and Basu (1980b) have considered this problem when the component lifetimes follow a Weibull distribution with unequal shape parameters. In this report we consider this problem when the components have a common, unknown shape parameter.

In section 2 we present the model we shall use for accelerated lifetests in the competing risk framework. In section 3 we obtain estimates of model parameters for type II and progressively censored data. In section 4 an illustrative example is presented.

2. THE MODEL

The problem considered in the sequel is as follows. Consider a p component system with component lifetimes $X_1, X_2, \ldots, X_p$. Suppose that under normal stress conditions these components have long lifetimes making testing at such conditions unfeasible. To reduce test time and cost, s stresses, $V_1, \ldots, V_s$ are selected and a life test is conducted at constant application of the selected stress. We wish to use this information to make inference about the component lifetimes under normal stress conditions.

Consider the following model introduced by Klein and Basu (1980b) elsewhere.

At a stress V_i, $i = 1, \ldots, s$ assume that the j^{th} component has a hazard rate given by

$$h_j(x, V_i;\ \alpha_j, \underline{\beta_j}) = g_j(x, \underline{\alpha_j})\lambda_j(V_i, \underline{\beta_j}) \tag{2.1}$$

$$i = 1, \ldots, s \quad j = 1, \ldots, p.$$

For $g_j(x, \alpha)$ a Weibull form is assumed, that is

$$g_j(x,\alpha) = \alpha t^{\alpha-1},\ \alpha \geq 0,\ t > 0, \tag{2.2}$$

where α is the same unknown constant for each component and this constant is independent of the stress V.

For $\lambda_j(V, \underline{\beta_j})$ we assume a model of the form

$$\lambda_{ij} = \lambda_j(V, \underline{\beta}_j) = \exp\left(\sum_{\ell=0}^{k_j} \beta_{j\ell}\theta_{j\ell}(V) \right) . \tag{2.3}$$

where $\theta_{j0}(V) = 1$ and $\theta_{j1}(V), \ldots, \theta_{jk_j}(V)$ are k_j non-decreasing functions of V. The $\theta_j(\cdot)$'s may differ from one component to another.

This model includes the standard models, namely, the power rule with $\lambda_j(V_j, \underline{\beta}_j) = \beta_{j0}V^{\beta_{j1}}$; The Arrhenius reaction rate model with $\lambda_j(V, \underline{\beta}_j) = \exp(\beta_{j0} - \beta_{j1}/V)$; and the Eyring model for a single stress with $\lambda_j(V, \underline{\beta}_j) = V^{\beta_{j1}} \exp(\beta_{j0} - \beta_{j2}/V)$ as special cases.

The model also can be derived from the interpretation of the effects of a carcinogen on a cell as proposed by Armitage and Doll (1961). To produce cancer in a single cell, k independent events must occur. The effect of an increased dose of a carcinogen is to increase the rate at which these k events occur. If, for the j^{th} disease, this increase is of the form $\exp(\beta_{j\ell}\theta_{j\ell}(V))$ for $\ell = 1, \ldots, k_j$ the model (2.3) is obtained. If this increase is assumed linear the model of Hartley and Sielkin (1977) is obtained. Thus the model of Hartley and Sielkin is a first order Taylor Series approximation to (2.3) when $\theta_{j\ell}(V) = V$ for $\ell = 1, \ldots, k_j$.

Consider an accelerated life test conducted at constant applications of s stress level, $V_1, \ldots, V_s$. Let $X_{i1}, X_{i2}, \ldots, X_{ip}$ denote the component lifetimes of the p component series system put on test at stress V_i. Assume that the component lifetimes are independent. We are not allowed to observe $X_{i1}, \ldots, X_{ip}$ directly but, instead, we observe $Y_i = \text{minimum}(X_{i1}, \ldots, X_{ip})$ and an indicator variable which describes which of the p components is the minimum. We shall use the method of maximum likelihood to estimate α and $\underline{\beta}_j = (\beta_{j0}, \ldots, \beta_{jk_j})$, $j = 1, \ldots, p$ for various censoring schemes.

3. ESTIMATION OF PARAMETERS

In this section we obtain maximum likelihood estimates for α and $\underline{\beta}_j$, $j = 1, \ldots, p$ for type II and progressively censored accelerated life tests. Since these estimates must be obtained numerically we also show how initialestimates can be obtained by a least square technique. We then obtain point and interval estimates for the component parameters at the use stress.

3.1 TYPE II CENSORING

For this censoring scheme n_i systems are put on test at each of the s stress levels and testing continues until a preassigned number r_i have failed at which time testing at that stress level is terminated. Let the ordered failure times of the r_i systems which fail be denoted by $Y_{i(1)}, Y_{i(2)}, \ldots, Y_{i(r_i)}$. Let r_{ij} denote the number of the r_i systems which failed due to failure of the j^{th} component $j = 1, \ldots, p$. The contribution to the total likelihood from the n_i systems on test at stress V_i is

$$L_i = \prod_{j=1}^{p} \lambda_{ij}^{r_{ij}} \exp(-T_i(\alpha)\lambda_{ij})\alpha^{r_i} \left(\prod_{\ell=1}^{r_i} Y_{i(\ell)}\right)^{\alpha-1} \tag{3.1.1}$$

where

$$T_i(\alpha) = \prod_{\ell=1}^{r_i} Y_{i(\ell)}^{\alpha} + (n_i - r_i)Y_{i(r_i)}^{\alpha}, \quad i = 1, \ldots, s \tag{3.1.2}$$

The overall likelihood is $L = \prod_{i=1}^{s} L_i$ so

$$L = \sum_{i=1}^{s} \sum_{j=1}^{p} r_{ij} \left(\sum_{\ell=0}^{k_j} \beta_{j\ell}\theta_{j\ell}(V_i)\right) - T_i(\alpha)\exp\left(\sum_{\ell=0}^{k_j} \beta_{j\ell}\theta_{j\ell}(V_i)\right)$$
$$+ r_i \ln\alpha + (\alpha-1)P_i \tag{3.1.3}$$

where

$$P_i = \sum_{\ell=1}^{r_i} \ln Y_{i\ell}.$$

The likelihood equations are

$$\frac{\delta \ln L}{\delta\alpha} = \sum_{i=1}^{s} -T_i^{(1)}(\alpha)\{\sum_{j=1}^{p} \exp(\sum_{\ell=1}^{k_j} \beta_{j\ell}\theta_{j\ell}(V_i))\} + \frac{r_i}{\alpha} + P_i = 0 \tag{3.1.4}$$

where

$$T_i^{(1)}(\alpha) = \sum_{\ell=1}^{r_i} Y_{i(\ell)}^{\alpha} \ln Y_{i(\ell)} + (n_i - r_i)Y_i^{\alpha} \ln Y_{i(r_i)}$$

$i = 1, \ldots, s$, and

$$\frac{\delta \ln L}{\delta\beta_{ju}} = 0 = \sum_{i=1}^{s} \theta_{ju}(V_i)[n_{ij} - T_i(\alpha)\exp(\sum_{\ell=0}^{k_j} \beta_{j\ell}\theta_{j\ell}(V_i))]$$

$$j = 1, \ldots, p \quad u = 1, \ldots, k_j \tag{3.1.5}$$

The solutions, $\hat{\alpha}$, $\hat{\beta}_{j\ell}$, $j = 1, \ldots, p$, $\ell = 1, \ldots, k_j$ of 3.1.4 and 3.1.5 are the respective maximum likelihood estimators.

The second partial derivative of L are

$$-\frac{\delta^2 \ln L}{\delta\alpha^2} = \sum_{i=1}^{s} \lambda_i T_i^{(2)}(\alpha) + \sum_{i=1}^{s} \frac{r_i}{\alpha^2} \text{ where} \tag{3.1.6}$$

$$T_i^{(2)}(\alpha) = \sum_{\ell=1}^{r_i} Y_{i(\ell)}^{\alpha}[\ln(Y_{i\ell})]^2 + (n_i - r_i)Y_{i(r_i)}^{\alpha}[\ln Y_{i(r_i)}]^2 ,$$

$i = 1, \ldots, s$.

$$-\frac{\delta^2 \ln L}{\delta\alpha\delta\beta_{ju}} = \sum_{i=1}^{s} T_i^{(1)}(\alpha)\theta_{jw}(V_i)\lambda_{ij} \quad j = 1, \ldots, p \quad u = 0, \ldots, k_j, \tag{3.1.7}$$

$$-\frac{\delta^2 \ln L}{\delta\beta_{ju}\delta\beta_{jw}} = \sum_{i=1}^{s} \lambda_{ij} T_i(\alpha)\theta_{jw}(V_i)\theta_{ju}(V_i) \tag{3.1.8}$$

$j = 1, \ldots, p \quad u = 0, \ldots, k_j \quad w = 0, \ldots, k_j$, and,

$$-\frac{\delta^2 \ln L}{\delta\beta_{ju}\delta\beta_{gw}} = 0 \quad j \neq g \quad j = 1, \ldots, p \qquad (3.1.9)$$

$$g = 1, \ldots, p \quad u = 0, \ldots, k_j \quad w = 0, \ldots, k_j .$$

To find $E(T_i(\alpha))$, $E(t_i^{(1)}(\alpha))$, $E(T_i^{(2)}(\alpha))$ note that the unordered $Y_{i\ell}$'s have a Weibull distribution with density

$$f(y_{i\ell}) = \alpha\lambda_i y_{i\ell}^{\alpha-1}\exp(-\lambda_i y_{i\ell}^{\alpha}),\ y_{i\ell} > 0, \quad i = 1, \ldots, s .$$

The m^{th} order statistic from a Weibull has density

$$f(y_{i(m)}) = \frac{n_i!}{(m-1)!(n_i-m)!}(\alpha\lambda_i)y_{i(m)}^{\alpha-1}[1-e^{-y_{i(m)}^{\alpha}}]^{m-1}$$

$$\exp(-(n_i-m+1)\lambda_i y_{i(m)}^{\alpha}), \quad 0 < y_{i(m)} < \infty. \qquad (3.1.10)$$

Now

$$E(Y_{i(m)}^{\alpha}) = \int_0^{\infty}\frac{n_i\,!}{(m-1)!(n_i-m)!}\,y^{\alpha}[(\lambda_i\alpha)y^{\alpha-1}][1-\exp(-y^{\alpha}\lambda_i)]^{m-1}$$

$$\exp(-(n_i-m+1)y^{\alpha}\lambda_i)dy$$

$$= \frac{n_i!}{(m-1)!(n_i-m)!\lambda_i}\int_0^{\infty}u[1-e^{-u}]^{m-1}\exp(-(n_i-m+1)u)du$$

$$= \frac{m}{\lambda_i}\binom{n_i}{m}\sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}\int_0^{\infty}ue^{-(n_i-m+1+k)}du$$

$$= \frac{m}{\lambda_i}\binom{n_i}{m}\sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}\frac{1}{(n_i-m+1+k)^2} .$$

Hence

$$E(T_i(\alpha)) = \sum_{m=1}^{r_i} \frac{m}{\lambda_i} \binom{n_i}{m} \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} \frac{1}{(n_i-m+k+1)^2}$$

$$+ (n_i-r_i) \frac{r_i}{\lambda_i} \binom{n_i}{r_i} \sum_{k=0}^{r_i-1} (-1)^k \binom{r_i-1}{k} \frac{1}{(n_i-m+k+1)^2}$$

$$= r_i/\lambda_i \tag{3.1.11}$$

Similarly,

$$E(T_i^{(1)}(\alpha)) = \frac{1}{\lambda_i\alpha} \{(1-\gamma- \quad \lambda_i) \cdot r_i + \sum_{m=1}^{r_i} m \binom{n_i}{m} \sum_{k=0}^{m-1} \frac{(-1)^k \binom{m-1}{k}}{(n_i-m+k+1)^2}$$

$$\cdot \ln(n_i-m+k+1) + (n_i-r_i) r_i \binom{n_i}{r_i} \sum_{k=0}^{m-1} \frac{(-1)^k \binom{r_i-1}{k}}{(n_i-r_i+k+1)^2}$$

$$\cdot \ln(n_i-m+r_i+1)\}. \tag{3.1.12}$$

where γ = .5772156649 is Euler's constant.

Also, by similar computations

$$E(T_i^{(2)}(\alpha)) = \frac{1}{\lambda_i\alpha^2} \{[\frac{\Pi^2}{6} - 1 + (1-\gamma) \quad - 2 \ln\lambda_i(1-\gamma)+(\ln\lambda_i)^2 r_i]$$

$$+ \sum_{m=1}^{r_i} m \binom{n_i}{m} \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} \cdot \frac{1}{(n_i-m+k+1)^2}$$

$$[(\ln^2(n_i-m+k+1) - 2(1-\gamma-\ln\lambda_i) \cdot \ln(n_i-m+k+1)]$$

$$+ r_i \binom{n_i}{r_i} \sum_{k=0}^{r_i-1} \frac{(-1)^k \binom{r_i-1}{k}}{(n_i-r_i+k+1)^2} [(\ln(n_i-r_i+k+1))^2$$

$$- 2(1-\gamma- \ln\lambda_i) \ln(n_i-r_i+k+1)]\}.$$

Let $a = -E(\frac{\delta^2 \ln L}{\delta\alpha^2})$. Let C_j be the column vector

$$C_j = \left[-E(\frac{\delta^2 \ln L}{\delta\alpha\delta\beta_{jo}}), \ \ldots, \ -E(\frac{\delta^2 \ln L}{\delta\alpha\delta\beta_{jk_j}})\right]^T, \ j = 1, \ldots, p, \text{ and}$$

let B_j be the k_j+1 by k_j+1 matrix whose elements are

$$b_{u+1,w+1} = -E(\frac{\delta^2 \ln L}{\delta\beta_{ju}\delta\beta_{jw}}) \quad u = 0, \ldots, k_j, \ w = 0, \ldots, k_j \ .$$

The inverse of the asymptotic correlation matrix is

$$\Sigma^{-1} = \begin{pmatrix} B & C \\ C^T & a \end{pmatrix} \quad \text{where } B = \begin{pmatrix} B_1 & & 0 \\ & B_2 \ddots & \\ 0 & & B_p \end{pmatrix} \tag{3.1.14}$$

and $C^T = (C_1^T, C_2^T, \ldots, C_p^T$. The value of Σ^{-1} is obtained by substituting 3.1.11, 3.1.12, and 3.1.13 in the appropriate places in 3.1.6, 3.1.7, and 3.1.8.

By a theorem of Rao (1973)

$$\Sigma = \begin{pmatrix} B^{-1}+FE^{-1}F^T & -FE^{-1} \\ -E^{-1}F^T & E^{-1} \end{pmatrix} \quad \text{where}$$

$$F = B^{-1}C, \text{ and } E = a-C^TB^{-1}C. \tag{3.1.15}$$

A consistant estimator of Σ is obtained by using $\hat{\alpha}$ and $\underline{\hat{\beta}}_j$, $j = 1, \ldots, p$ in the appropriate expressions 3.1.11, 3.1.12, and 3.1.13.

3.2 TYPE I PROGRESSIVE CENSORING

For this censoring scheme N_i items are put on test at stress V_i. Let $\tau_{i1}, \tau_{i2}, \ldots, \tau_{iM_i}$ be fixed censoring times. At censoring time $\tau_{i\ell}$, $\ell = 1, \ldots, M_i-1$, a fixed number $c_{i\ell}$, items are removed from the study. At time τ_{iM_i} either a fixed number c_{iM_i} items are removed from the test and testing continues until all remaining items have failed or the test is terminated with a random number c_{iM_i} items still functioning. This test scheme has the advantage of allowing for some items with extremely long lifetimes to be encluded in the study. Clearly the usual type I fixed time censoring is a special case of this consoring scheme with $M_i = 1$ and c_{i1} random. We assume that N_i is sufficiently large so that at least $c_{i\ell}$, $\ell = 1, \ldots, M_{i-1}, M_i$ items are still functioning to be censored.

Let $n_i = N_i - \sum_{i=1}^{M_i} c_{ik}$ be the number of systems which failed and let $Y_{i1}, Y_{i2}, \ldots, Y_{in_i}$ denote the failure times. Let r_{ij} denote the number of systems which fail from cause j at stress V_i, $j = 1, \ldots, p$, $i = 1, \ldots, s$. The N_i items on test at stress V_i, $i = 1, \ldots, s$ contribute

$$L_i = \prod_{j=1}^{p} \lambda_{ij}^{r_{ij}} \exp(-\lambda_{ij} T_i(\alpha)) \alpha^{n_i} \left(\prod_{\ell=1}^{n_i} Y_{i\ell} \right)^{-1} \tag{3.2.1}$$

to the total likelihood where

$$T_i(\alpha) = \sum_{\ell=1}^{n_i} Y_{i\ell}^{\alpha} + \sum_{\ell=1}^{M_i} \tau_{i\ell}^{\alpha} c_{i\ell} \tag{3.2.2}$$

Again the total log likelihood is given by 3.1.3 with $T_i(\alpha)$ as in 3.2.2 and $P_i = \sum_{\ell=1}^{n_i} \ln Y_i$. Maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}_j$, $j = 1, \ldots, p$ of α and β_j are obtained by solving numerically 3.1.4 and 3.1.5 with

$$T_i^{(1)}(\alpha) = \sum_{\ell=1}^{n_i} Y_{i\ell}^{\alpha} \ln Y_{i\ell} + \sum_{\ell=1}^{M_i} \tau_{i\ell}^{\alpha} \ln \tau_{i\ell} c_{i\ell}. \tag{3.2.3}$$

The second partial derivatives of L are given by 3.1.6, 3.1.7, 3.1.8, and 3.1.9 with

$$T_i^{(2)} = \sum_{\ell=1}^{n_i} Y_{i\ell}^{\alpha}(\ln Y_{i\ell})^2 + \sum_{\ell=1}^{M_i} c_{i\ell}\tau_{i\ell}^{\alpha}(\ln \tau_{i\ell})^2 \tag{3.2.4}$$

We now calculate $E(T_i(\alpha))$, $E(T_i^{(1)}(\alpha))$, and $E(T_i^{(2)}(\alpha))$.

Consider any of the s-stress levels. For notational convenience we shall suppress the subscript i. Let N items be put on test. Let $0 = \tau_o < \ldots < \tau_M < \tau_{M+1} = \infty$ be censoring points. At time τ_ℓ, $\ell = 1, \ldots, M$, c_ℓ items are removed from test. For $\ell = 1, \ldots, M-1$, c_ℓ is a fixed constant while c_M is either random or fixed depending on if testing is terminated at τ_M or not. Let $n = N - \sum_{\ell=1}^{M} c_\ell$ be the number of items which are observed to fail.

Let $Y_1, \ldots, Y_n$ denote the failure times of the n failures. For $\ell = 1, \ldots, n$, $Y_\ell = \min(X_{1\ell}, \ldots, X_{p\ell})$ where $X_{j\ell}$ is the failure time of the j^{th} component of the ℓ^{th} item which fails. By assumption the Y_ℓ's have a Weibull distribution with survival function

$$\overline{F}(y) = \exp(-\lambda y^{\alpha}),\ y \geq 0,\ \alpha,\ \lambda > 0 \tag{3.2.5}$$

Let f_ℓ denote the number of failures in the interval $[\tau_{\ell-1}, \tau_\ell)$, $i = 1, \ldots, M+1$. Let $U_{\ell k}$, $k = 1, \ldots, f_\ell$, $\ell = 1, \ldots, M+1$ denote the failure times of those f_ℓ items which fail in the interval. Let

$$F_\ell = P(Y < \tau_\ell) = 1 - \exp(-\lambda\tau_\ell^{\alpha}) \quad i = 1, \ldots, M+1 \tag{3.2.6}$$

denote the probability an item fails before time τ_ℓ, and let $\overline{F}_\ell = 1 - F_\ell$.

Cohen (1963) shows that for c_M a fixed constant

$$E(f_\ell) = \begin{cases} NF_\ell & \text{for } \ell = 1 \\ (N - \sum_{k=1}^{\ell-1} \frac{c_k}{\overline{F}_k})(F_\ell - F_{\ell-1}) & \text{for } \ell = 1, \ldots, M+1 \end{cases} \tag{3.2.7}$$

For c_M random 3.2.7 holds for $\ell = 1, \ldots, M$.

Now for $\ell = 1, \ldots, M$ let $S = \sum_{j=1}^{f_\ell} U_{\ell j}^{\alpha}$ and note that

$$E(S_\ell) = EE\left(\sum_{j=1}^{f_\ell} U_{\ell j}^{\alpha} \middle| f_\ell\right)$$

$$= E(f_\ell)E(U_{\ell j}^{\alpha} | \tau_{\ell-1} < U_{\ell 1} < \tau_\ell) \ .$$

The conditional density of $U_{\ell j}$ given $U_{\ell j} \varepsilon (\tau_{\ell-1}, \tau_\ell)$ is

$$f_{U\ell}(u) = \begin{cases} \dfrac{\alpha\lambda u^{\alpha-1}\exp(-\lambda u^{\alpha})}{F_\ell - F_{\ell-1}} & \text{if } \tau_{\ell-1} < u < \tau_\ell \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$E(U_{\ell j}^{\alpha} | \tau_{\ell-1} < U_\ell < \tau_\ell) = \frac{1}{\lambda}\{[\lambda\tau_{\ell-1}^{\alpha}+1)\overline{F}_{\ell-1} - (\lambda\tau_\ell^{\alpha}+1)\overline{F}_\ell]/(F_\ell - F_{\ell-1})\}$$

so

$$E(S_\ell) = \frac{1}{\lambda}(N - \sum_{j=1}^{\ell-1} \frac{c_j}{\overline{F}_j})((\lambda\tau_{\ell-1}^{\alpha}+1)\overline{F}_{\ell-1} - (\lambda\tau_\ell^{\alpha}+1)\overline{F}_\ell)\ell = 2, \ldots, M \tag{3.2.8}$$

and

$$E(S_\ell) = \frac{1}{\lambda} N(1 - (\lambda\tau_1^{\alpha}+1)\overline{F}_1), \quad \ell = 1 \ .$$

We now consider separately the two possible cases for c_M.

__Case 1__. c_M fixed.

For this case note that 3.4.8 holds for $\ell = M + 1$. So

$$\lambda\sum_{\ell=1}^{M+1} E(S_\ell) = N \sum_{\ell=1}^{M+1}(\lambda\tau_{\ell-1}^{\alpha}+1)\overline{F}_{\ell-1} - (\lambda\tau_\ell^{\alpha}+1)\overline{F}_\ell$$

$$- \sum_{\ell=1}^{M} \frac{c_\ell}{\overline{F}_\ell} \sum_{j=1}^{M+1} [(\lambda\tau_{j-1}^{\alpha}+1)\overline{F}_{j-1} - (\lambda\tau_j+1)\overline{F}_j] \ .$$

since $\tau_o = 0$, $\overline{F}_0 = 1$ and $\overline{F}_{M+1} = 0$ we have

$$\sum_{\ell=1}^{M+1} E(S_\ell) = \frac{1}{\lambda}\{N - \sum_{\ell=1}^{M} c_\ell(\lambda\tau_\ell^\alpha+1)\} .$$

Thus $E(T_i(\alpha)) = \dfrac{N_i - \sum_{\ell=1}^{M_i} c_{i\ell}}{\lambda_i}$, $i = 1, \ldots, s.$ (3.2.9)

Similarly,

$$E(T_i^{(1)}(\alpha)) = \frac{1}{\lambda_i\alpha}\{N_i(1 - \gamma - \ell n\,\lambda_i)$$

$$- \sum_{\ell=1}^{M_i} c_{i\ell}[\int_{\lambda_i\tau_{i\ell}^\alpha}^{\infty} u\,\ell n u\,\exp(-u - \lambda_i\tau_{i\ell}^\alpha)du$$

$$- \ell n\lambda_i - \lambda_i\tau_{i\ell}^\alpha\,\ell n\,\lambda_i\tau_{i\ell}^\alpha]$$, where γ is Euler's constant.

The integral must be evaluated numerically.

Also by similar computations

$$E(T_i^{(2)}(\alpha)) = \frac{1}{\alpha^2\lambda_i}\quad N_i(\frac{\Pi^2}{6} - 2\gamma + \gamma^2) - 2\ell n\,\lambda_i(1-\gamma) + (\ell n\,\lambda_i)^2$$

$$- \sum_{\ell=1}^{M_i} c_{i\ell}[\int_{\lambda_i\tau_{i\ell}^\alpha}^{\infty} (\ell n\,u)^2 u e^{-(u-\lambda_i\tau_{i\ell}^\alpha)} du$$

$$- 2\,\ell n\lambda_i \int_{\lambda_i\tau_{i\ell}^\alpha}^{\infty} (\ell n u) u e^{-(u-\lambda_i\tau_{i\ell}^\alpha)} du + \ell n\,\lambda_i$$

$$+ \lambda_i\tau_{i\ell}^\alpha(\ell n\,\lambda_i\tau_{i\ell}^\alpha)(\ell n(\lambda_i/\tau_{i\ell}^\alpha))]\}$$

Case II. c_M random.

For this scheme all testing stops at time τ_M with c_M items still functioning. Clearly

$$c_M = N - \sum_{\ell=1}^{M-1} c_\ell - \sum_{\ell=1}^{M} f_\ell .$$

So from 3.4.7

$$E(c_M) = \overline{F}_M \left(N - \sum_{\ell=1}^{M} \frac{c_\ell}{\overline{F}_\ell}\right) . \qquad (3.2.12)$$

Thus

$$E\left(\sum_{i=1}^{M} S_\ell\right) = \frac{1}{\lambda\alpha}\{N(1-(\lambda\tau_M^\alpha+1)\overline{F}_M - \sum_{\ell=1}^{M-1} \frac{c_\ell}{\overline{F}_\ell} [(\lambda\tau_\ell^\alpha+1)\overline{F}_\ell - (\lambda\tau_M^\alpha+1)\overline{F}_M]\}$$

Hence

$$E(T_i(\alpha)) = \frac{1}{\lambda_i}\{N_i F_{iM_i} - \sum_{\ell=1}^{M_i-1} r_{i\ell}(1 - \frac{\overline{F}_{i\ell}}{\overline{F}_{iM_i}})\} \qquad (3.2.13)$$

Similarly,

$$E(T_i^{(1)}(\alpha)) = \frac{1}{\lambda_i\alpha}\{N_i[\int_0^{\lambda_i\tau_{iM_i}^\alpha} u\,\ell n\, u e^{-u}du - \ell n\lambda_i F_{iM_i} + \overline{F}_{iM_i}\tau_{iM_i}^\alpha\, \ell n(\lambda_i\tau_{iM_i}^\alpha)]$$

$$- \sum_{\ell=1}^{M_i-1} c_{i\ell}\left(\int_{\lambda_i\tau_{i\ell}^\alpha}^{\lambda_i\tau_{iM_i}^\alpha} u\,\ell n u e^{-u}\cdot\frac{1}{\overline{F}_i} - \ell n\lambda_i(1-\overline{F}_{iM_i}/\overline{F}_{i\ell})\right.$$

$$\left. - \lambda_i\tau_{i\ell}^\alpha\,\ell n(\lambda_i\tau_{i\ell}^\alpha)+\lambda_i\tau_{iM_i}^\alpha\,\ell n(\lambda_i\tau_{iM_i}^\alpha)\overline{F}_{iM_i}/\overline{F}_{i\ell}\right)\}, \qquad (3.4.14)$$

and

$$E(T_i^{(2)}(\alpha)) = \frac{1}{\lambda_i\alpha^2} N_i[\int_0^{\lambda_i\tau_{iM_i}^\alpha} u(\ell n u)^2 e^{-u}du - 2\,\ell n\lambda_i \int_0^{\lambda_i\tau_{iM_i}^\alpha} u\,\ell n u e^{-u}du$$

$$+(\ell n\,\lambda_i)^2 F_{iM_i} + [(\ell n\,\lambda_i)^2 - (\ell n\,\tau_{iM_i}^\alpha)^2]\lambda_i\tau_{iM_i}^\alpha\overline{F}_{iM_i}]$$

$$- \sum_{\ell=1}^{M_i-1} c_{i\ell}[\int_{\lambda_i\tau_{i\ell}^\alpha}^{\lambda_i\tau_{iM_i}^\alpha} \frac{u(\ell n\, u)^2}{\overline{F}_{i\ell}} e^{-u}du - 2\ell n\,\lambda_i \int_{\lambda_i\tau_{i\ell}^\alpha}^{\lambda_i\tau_{iM_i}^\alpha} u^{\ell n}\, u e^{-u}du$$

$$+(\ln\lambda_i)^2(1-\overline{F}_{iM_i}/\overline{F}_{i\ell})+\lambda_i\tau_{i\ell}^{\alpha}[(\ln\lambda_i)^2-(\ln\tau_{i\ell}^{\alpha})^2]$$

$$-\lambda_i\tau_{iM_i}^{\alpha}\overline{F}_{iM_i}/\overline{F}_{i\ell}[(\ln\lambda_i)^2-(\ln\tau_{iM_i}^{\alpha})^2]]\} \ . \qquad (3.2.15)$$

The asymptotic covariance matrix is obtained by making the appropriate substitution in 3.1.6, 3.1.7 and 3.1.8.

3.3 INITIAL SOLUTIONS TO LIKELIHOOD EQUATIONS

To solve the likelihood equation numerically initial estimates of the parameters which are close to the maximum likelihood estimates are needed. To obtain such estimates we shall first obtain an estimate of α, then transform the data to exponential observations and apply a least squares technique.

Consider any one of the s stress levels, v_i. Let $\hat{\alpha}_i$ be an estimate of α based on observations at this stress level only. Such estimates can be obtained by using techniques described in Mann, Schaffer and Singpurwalla (1974) or by graphical methods described in Nelson (1972). These estimates are then pooled to obtain an estimate $\tilde{\alpha}$ of α. If the $\hat{\alpha}_i$'s differ too much from one stress level to the next this will cast doubt on the assumption of equal shape parameters.

To obtain estimators of the $\underline{\beta}$'s we first make the transformation $W_{i\ell} = Y_{i\ell}^{1/\tilde{\alpha}}$, $i = 1, \ldots, s$, $\ell = 1, \ldots, n_i$. If $\tilde{\alpha}$ is equal to the true α then $W_{i\ell}$ will have an exponential distribution with hazard rate λ_i.

Let $T_i(\tilde{\alpha})$ be defined by 3.1.1 or 3.2.1. One can show that, using information collected at a single stress level only, the maximum likelihood estimator of λ_{ij} is

$$\ln(r_{ij}/T_i(\tilde{\alpha})) \qquad (3.3.1)$$

say. For sufficiently large n

$$E(\hat{\eta}_i) \simeq \ln\lambda_{ij} = \sum_{\ell=0}^{k_j} \beta_{j\ell}\theta_{j\ell}(v_i)$$

$$\text{and } \mathrm{Var}(\hat{\eta}_i) \stackrel{\sim}{=} \lambda_{ij} E(T_i(\tilde{\alpha})) \tag{3.3.2}$$

Least squares estimates of the $\beta_{j\ell}$'s are obtained by using weighted least squares as described in Draper and Smith (1966, p. 77-81). Let $\underline{\eta}$ be the sX1 column vector of the $\hat{\eta}_i$'s. Let θ be the $s \times k_j + 1$ matrix defined by

$$\theta = (\theta_{j\ell}(V_i)), \quad \ell = 0, \ldots, k_j \quad i = 1, \ldots, s$$

and let B be the column vector $(\beta_{j0}, \ldots, \beta_{jk_j})^T$. The model of interest is

$$\underline{\eta} = \theta B + \varepsilon \tag{3.3.3}$$

where $E(\varepsilon) = 0$ and $\mathrm{Var}(\varepsilon) = A$ where A is the diagonal matrix with elements $\mathrm{Var}(\eta_i)$ along the main diagonal. The weighted least squares estimators of $\underline{\beta}_j$ are

$$\tilde{\underline{\beta}}_j = (\theta^T A^{-1} \theta)^{-1} \theta^T A^{-1} \eta \; . \tag{3.3.4}$$

The variance of these estimators is

$$\mathrm{Var}(\tilde{\underline{\beta}}_j) = (\theta^T A^{-1} \theta)^{-1} \; . \tag{3.3.5}$$

3.4. ESTIMATION OF USE STRESS PARAMETER

Suppose an accelerated lifetest as been conducted according to one of the censoring schemes discussed in sections 3.1 or 3.2. Let $\hat{\underline{\beta}}_j = (\hat{\beta}_o, \ldots, \hat{\beta}_{k_j})$ and $\hat{\alpha}$ be the maximum likelihood estimators of $\underline{\beta}_j$ and α respectively. Let $\Sigma_{jj}(\hat{\Sigma}_{jj})$ be the (estimated) asympotic variance-covariance matrix of $\underline{\beta}_j$. Let $\Sigma_{j\alpha}(\hat{\Sigma}_{j\alpha})$ be the $k_j + 1 \times 1$ column vector whose ℓ element is the (estimated) asymptotic covariance of $\hat{\beta}_{j\ell}$ and $\hat{\alpha}$, $\ell = 0, \ldots, k_j$, $j = 1, \ldots, p$. Let $\sigma^2_{\alpha\alpha}(\hat{\sigma}^2_{\alpha\alpha})$ be the (estimated) asymptotic variance of α. Let 0 be the matrix whose elements are all zero, then the asymptotic variance matrix of $(\hat{\underline{\beta}}_1, \hat{\underline{\beta}}_2, \ldots, \hat{\underline{\beta}}_p, \alpha)$ is of the form

$$\Sigma = \begin{bmatrix} \Sigma_{11} & 0 & 0 & 0 & \Sigma_{1\alpha} \\ 0 & \Sigma_{12} & 0 & 0 & \Sigma_{2\alpha} \\ 0 & 0 & & & \\ 0 & 0 & & \Sigma_{pp} & \Sigma_{\alpha} \\ \Sigma_{1\alpha}^T & \Sigma_{\alpha\alpha}^T & & \Sigma_{p\alpha}^T & \sigma_{\alpha\alpha}^2 \end{bmatrix} \qquad (3.4.1)$$

Let $\hat{\Sigma}$ denote the estimated value of $\hat{\Sigma}$.

We shall use this information to make inference about component life under some use stress, V_u.

Recall that the scale parameter of the time to failure distribution is given by

$$\lambda_{ju} = \exp\left(\sum_{\ell=0}^{k_j} \beta_{j\ell}\theta_{j\ell}(V_u)\right), \quad j = 1, \ldots, p . \qquad (3.4.2)$$

at the use stress. The maximum likelihood estimator of λ_{ju} is given by

$$\hat{\lambda}_{ju} = \exp\left(\sum_{\ell=0}^{k_j} \hat{\beta}_{j\ell}\theta_{j\ell}(V_u)\right), \quad j = 1, \ldots, p . \qquad (3.4.3)$$

This estimator is biased by a factor of $\exp(\sigma_{ju}^2/2)$ where

$$\sigma_{ju}^2 = (1, \theta_{j1}(V_u), \ldots, \theta_{jk_j}(V_u)\Sigma_{jj}(1, \theta_{j1}(V_u), \ldots, \theta_{jk_j}(V_u))^T. \qquad (3.4.4)$$

An unbiased estimator of λ_{ju} is given by

$$\tilde{\lambda}_{ju} = \hat{\lambda}_{ju} \exp(-\sigma_{ju}^2/2), \quad j = 1, \ldots, p . \qquad (3.4.5)$$

Asymptotic $(1 - \gamma) \times 100\%$ confidence intervals for λ_{ju} are given

$$(\hat{\lambda}_{ju}\exp(-Z_{1-\gamma/2}\hat{\sigma}_{ju}), \hat{\lambda}_{ju}\exp(Z_{1-\gamma/2}\hat{\sigma}_{ju}) \qquad (3.4.6)$$

where $Z_{1-\gamma/2}$ is the $1-\gamma/2^{th}$ percentage point of a standard normal random variable.

Consider now the cummulative hazard rate of the time to failure distribution of the j^{th} component which is, at time $t > 0$ and stress V_u, given by

$$\Lambda_{ju}(t) = t^{\alpha}\exp\left(\sum_{\ell=0}^{k_j} \beta_{j\ell}\theta_{j\ell}(V_j)\right), \quad j = 1, \ldots, p. \quad (3.4.7)$$

The maximum likelihood estimator of $\Lambda_{ju}(t)$ is, by the invariance property,

$$\hat{\Lambda}_{ju}(t) = t^{\hat{\alpha}}\exp\left(\sum_{\ell=0}^{k_j} \hat{\beta}_{j\ell}\theta_{j\ell}(V_u)\right), \quad j = 1, \ldots, p. \quad (3.4.8)$$

Now, asymptotically $\ln\hat{\Lambda}_{ju}(t)$ has a normal distribution with mean $\Lambda_{ju}(t)$ and variance

$$\sigma_{ju}^2(t) = (1, \theta_{j1}(V_u), \ldots, \theta_{jk_j}(V_u), \ln(t)) \begin{pmatrix} \Sigma_{jj} & \Sigma_{j\alpha} \\ \Sigma_{j\alpha}^T & \Sigma_{\alpha\alpha}^2 \end{pmatrix} \begin{pmatrix} \theta_{j1}^1(V_u) \\ \\ \theta_{jk_j}(V_u) \\ \ln(t) \end{pmatrix}$$

$$j = 1, \ldots, p, \; t > 0 \quad (3.4.9)$$

Hence $\hat{\Lambda}_{ju}(t)$ is a biased estimator of $\Lambda_{ju}(t)$. A reduced bias estimator of $\Lambda_{ju}(t)$ is given by

$$\tilde{\Lambda}_{ju}(t) = \tilde{\Lambda}_{ju}(t)\exp(\hat{\sigma}_{ju}(t)/2) \; j = 1, \ldots, p, \; t > 0 \quad (3.4.10)$$

which also has reduced mean squared error

A $(1-\gamma)\times 100\%$ confidence interval for $\Lambda_{ju}(t)$ is given by

$$(\hat{\Lambda}_{ju}(t)\exp(-Z_{1-\gamma/2}\hat{\sigma}_{ju}(t)), \hat{\Lambda}_{ju}(t)\exp(Z_{1-\gamma/2}\hat{\sigma}_{ju}(t))) .$$

Let $\overline{F}_{ju}(t) = \exp(-\Lambda_{ju}(t))$, $t > 0$ be the survival function of the j^{th} component. The maximum likelihood estimator of $\overline{F}_{ju}(t)$ is

$$\hat{\overline{F}}_{ju}(t) = \exp(-\hat{\Lambda}_{ju}(t)) \quad j = 1, \ldots, p, \; t > 0 \quad (3.4.12)$$

Approximate $(1-\gamma)\times 100\%$ confidence intervals for $\bar{F}_{ju}(t)$ are given by

$$(\hat{\bar{F}}_{ju}(t)^{\exp(Z_{1-\gamma/2}\hat{\sigma}_{ju})}, \hat{\bar{F}}_{ju}(t)^{\exp(-Z_{1-\gamma/2}\hat{\sigma}_{ju})}), \ j = 1, \ldots, p \tag{3.4.13}$$

3.5. DEPENDENT RISKS

The assumption of independent causes of failure may be relaxed to include a class of Weibull distributions with dependent causes of failure. To illustrate how this may be done we shall consider the bivariate case with the obvious extension to more than two risks.

Let X_1, X_2 denote the failure times of the two components in a series system. Let U_1 be the time until the system fails due to failure of the first component alone, U_2 the time until failure from the second component alone, and, U_{12} the time until simultaneous failure of both components. At a stress V assume that U_1, U_2, U_{12} are independent Weibull random variables with shape parameter α and scale parameter $\lambda_j(V, \underline{\beta}_j)$, $j = 1, 2, 12$, given by 2.1.2. Clearly $X_1 = \min(U_1, U_{12})$ and $X_2 = \min(U_2, U_{12})$. X_1 and X_2 are both distributed Weibull with shape parameter α and scale parameter $\lambda_1(V, \underline{\beta}_1) + \lambda_{12}(V, \underline{\beta}_{12})$, $\lambda_2(V, \underline{\beta}_2) + \lambda_{12}(V, \underline{\beta}_{12})$, respectively. The joint survival function of (X_1, X_2) is given by

$$\bar{F}(x_1, x_2) = \exp(-\lambda_1(V, \underline{\beta}_1)x_1 - \lambda_2(V, \underline{\beta}_2)x_2 - \lambda_{12}(V, \underline{\beta}_{12})\max(x_1, x_2)\alpha).$$

This is the bivariate Weibull distribution proposed by Lee and Thompson (1974).

To estimate the parameters we perform an accelerated life test as described in the previous sections. The failure causes are now failure from the first component alone, the second component alone, and, simultaneous failure from both components. Estimators of $\underline{\beta}_1$, $\underline{\beta}_2$, $\underline{\beta}_{12}$, and α can be obtained as before. Estimates of the parameters of X_1 and X_2 now follow in a straight forward manner.

4. EXAMPLE

As an example of these procedures we shall consider an example given in Nelson (1974a). The problem is to analyze an accelerated life test conducted on Class-H insulation systems for electric motors. There are three possible types of insulation failures corresponding to distinct parts of the insulation system, namely Turn, Phase, and Ground. The failure cause is determined by an engineering examination of the failed motor.

The purpose of the experiment is to estimate the average life of such insulation systems at a design temperature of 180°C. A median life of 20,000 hours is necessary for the satisfactory performance of these insulation systems. To reduce test time and cost an accelerated life test was conducted at 4 accelerated temperatures, namely, 190° C, 220° C, 240° C, and 260° C.

The accelerated life test was conducted by putting 10 motors on test at each of the 4 stress levels. Motors were run until they failed, then the cause of failure was found and isolated and motors were run until a second failure occurred. The results of this study are reported in Nelson (1974a). The data followed a $\log_{10}$ normal distribution so the Weibull theory results do not apply.

To illustrate the results of the previous section Nelson's example is reporduced by simulating the life test using a Weibull model with shape parameter 1 for each failure cause. The shift parameters are chosen by fitting an Arrhenius Reaction Rate model to the estimated component medians obtained by Nelson. The model is

$$\lambda_i(V;\ \underline{\beta}_j) = \exp(\beta_{j0} + \beta_{j1}\theta_{j1}(V)), \quad j = 1, 2, 3 \qquad (4.1)$$

where $\theta_{i1}(V) = -1000/V$ for $j = 1, 2, 3$ and V is the temperature in degrees absolute. The absolute temperature is 273.16 plus the centrigrade temperature. The values of (β_{j0}, β_{j1}), $j = 1$, 2, 3 are as follows:

Table 4.2 True Values of β_0, β_1

	β_0	β_1
Turn	8.2607	8.0106
Phase	3.7748	6.1253
Ground	13.0340	10.6487

Twenty Weibull observations were generated at each of the four stress levels. The data are in Table 4.1.

Using the data at each stress level only the two order statistic estimates of α discussed in Derbey (1966) are $\hat{\alpha}_1 = 2.224$, $\hat{\alpha}_2 = 1.103$, $\hat{\alpha}_3 = 1.260$, and $\hat{\alpha}_4 = 1.155$. This suggests an initial estimate of $\tilde{\alpha} = 1.425$ for α.

The data is now transformed by letting $W_{i\ell} = Y_{i\ell}^{1/1.425}$, $i = 1, \ldots, s$, $\ell = 1, \ldots, n_i$ and the least squares procedure of section 3.3 is applied. The resulting initial estimate of the β's are as follows

$$\tilde{\beta}_{TURN} = (4.3136,\ 4.9144),\ \tilde{\beta}_{PHASE} = (1.5206,\ 3.8801)\ ,$$

and

$$\tilde{\beta}_{GROUND} = (6.1364,\ 6.1194)\ .$$

The maximum likelihood estimates are obtained by using a two stage Newton-Raphson procedure. In the first stage the likelihood in maximized with respect to α using the β's obtained in the previous stage. In the second stage, using this α, the likelihood is maximized with respect to the β's. The procedure is terminated when the relative increase in the likelihood is no more than .0001.

In this case the two stage procedure terminated after 19 steps. The maximum likelihood estimators are $\hat{\alpha} = 1.0995$, $\hat{\underline{\beta}}_{TURN} = (6.6142,\ 7.5033)$, $\hat{\underline{\beta}}_{PHASE} = (4.1230,\ 6.6608)$, and $\hat{\underline{\beta}}_{GROUND} = (8.2946,\ 8.6483)$. The estimated covariance matrix of $(\hat{\underline{\beta}}_{TURN}, \hat{\underline{\beta}}_{PHASE}, \hat{\underline{\beta}}_{GROUND}, \hat{\alpha})$, computed from the results of section 3.1 with $r_i = n_i$, is

[redacted]

Motor	180 Degrees Failure Time	180 Degrees Cause	190 Degrees Failure Time	190 Degrees Cause	220 Degrees Failure Time	220 Degrees Cause	240 Degrees Failure Time	240 Degrees Cause
1	5606.0781	Turn	1628.8145	Ground	344.1240	Phase	557.4395	Ground
2	4905.0859	Turn	1097.6609	Turn	761.8518	Phase	156.9030	Phase
3	2871.9370	Phase	630.0374	Phase	1562.7520	Turn	906.5212	Phase
4	2762.9712	Ground	1520.8772	Ground	276.9924	Ground	61.1210	Turn
5	3413.8027	Turn	708.5212	Phase	482.2432	Turn	773.3906	Turn
6	6321.7617	Turn	205.9655	Phase	213.3295	Turn	148.7977	Ground
7	4847.3906	Turn	185.6579	Turn	1434.3723	Turn	41.1974	Turn
8	2690.2847	Turn	434.2930	Turn	1486.6152	Turn	787.6323	Turn
9	38.9871	Phase	1938.7297	Phase	1355.4917	Turn	224.2534	Ground
10	2358.2275	Phase	3093.8237	Turn	1374.0374	Phase	405.3303	Ground
11	3755.4043	Ground	1171.8782	Ground	725.5413	Turn	1071.6702	Ground
12	4898.8477	Phase	1108.7510	Ground	917.9756	Ground	407.1978	Turn
13	3900.2310	Turn	27.5321	Turn	2970.2925	Ground	306.0037	Ground
14	1196.4922	Turn	1428.3220	Turn	609.9128	Turn	422.7825	Turn
15	6000.6875	Ground	263.7917	Phase	89.8835	Turn	178.5943	Turn
16	1645.1550	Ground	1113.6123	Turn	741.6179	Turn	588.4976	Turn
17	4021.5698	Ground	965.0088	Ground	706.0217	Ground	301.6204	Phase
18	2643.6931	Turn	49.1324	Phase	347.2078	Turn	14.8288	Turn
19	4760.6289	Turn	350.6594	Turn	238.5782	Phase	1315.0032	Ground
20	1621.5481	Turn	2026.7441	Phase	1001.3643	Ground	90.0674	Turn

$$\Sigma = \begin{bmatrix} 8.615 & 4.285 & .0014 & 0.0 & .0014 & 0.0 & -.0035 \\ 4.285 & 2.138 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ .0014 & 0.0 & 25.387 & 12.577 & .0014 & 0.0 & -.0035 \\ 0.0 & 0.0 & 12.577 & 6.245 & 0.0 & 0.0 & 0.0 \\ .0014 & 0.0 & .0014 & 0.0 & 15.894 & 7.959 & -.0035 \\ 0.0 & 0.0 & 0.0 & 0.0 & 7.959 & 3.997 & 0.0 \\ -.0035 & 0.0 & -.00353 & 0.0 & -.0035 & 0.0 & .0092 \end{bmatrix}$$

At the use stress of 180°C the estimates of component survival at a mission time of 20,000 hours are .0763 for turn failures. 90% confidence intervals for components survival at 20,000 hours are

(.0013, .3688) for turn failures,

(.0004, .7872) for phase failures, and

(.00242, .8168) for ground failures.

The maximum likelihood estimates of the scale parameters at 180°C are

$\hat{\lambda}_{TURN} = .0000480$, $\hat{\lambda}_{PHASE} = .0000255$, and $\hat{\lambda}_{GROUND} = .0000206$.

The reduced bias estimates are

$\hat{\lambda}_{TURN} = .0000454$, $\hat{\lambda}_{PHASE} = .0000221$, and $\hat{\lambda}_{GROUND} = .0000184$.

90% confidence intervals for the shape parameters are

(.0000275, .0000839) for turn failures,

(.0000105, .0000619) for phase failures, and

(.0000093, .0000455) for ground failures.

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